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Global Existence and Stability of Solutions for Reaction Diffusion Functional Differential Equations

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In this article, a class of reaction diffusion functional differential equations is investigated. The global existence and uniqueness of solutions and the stability of the trivial solution are obtained. Some applications are also discussed. The method proposed in this article is a combination of the iterative analysis, the fixed point theorem, and the comparison principle.

1. INTRODUCTION

Reaction diffusion functional differential equations, also named parabolic functional differential equations, are a new important branch of partial differential equations. The related work has been increasing gradually in present days. Lots of work stressed the existence of solutions, but few results have been reported on the stability problem. This paper studies a class of parabolic functional differential systems with time delay. By using the comparison principles, we establish the existence and uniqueness of solutions and solve the stability problem for the trivial solution at the same time. The method employed in this paper is a generalization of the one proposed by A. Stokes [7] to reaction diffusion functional differential equations. Some results obtained in this article are similar to the recent work of Travis and Webb [8–10] on abstract functional differential equations. This paper consists of two major parts: Section 2 studies the global existence, uniqueness of solutions, and the stability of the trivial solution. Section 3 applies the obtained results to two known problems which have

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been discussed by A. Pazy in [6] and A. Inoue in [4], respectively. The notations used in this paper follows those used by Hale [3] and Travis and Webb [8].

Let X be a real Banach space, $C = C([-r, 0], X)$. For $\varphi \in C$, we denote $\|\varphi\|_C = \sup_{-r \le \theta \le 0} \|\varphi\|_X$. We consider the following initial value problem on X,

$$\begin{cases} \dfrac{du}{dt} = Au + f(t, u_t), & t \ge 0, \\ u = \varphi(t), & t \in [-r, 0], \end{cases} \tag{1.1}$$

where A is a linear operator on X, $u_t = u(t + \theta)$, $-r \le \theta \le 0$ $(r > 0)$.

We introduce the following basic assumptions:

(H_1) the linear operator A is an infinitesimal generator of a strongly continuous semigroup $T(t)$ on X;

(H_2) f: $R_+ \times C \to X$ is continuous, $f(t, 0) = 0$, and it satisfies the local Lipschitz condition with respect to φ, i.e., for $\varphi_1, \varphi_2 \in C$, the following expression holds

$$\| f(t, \varphi_1) - f(t, \varphi_2) \|_X \le L(t, \|\varphi_1\|_C, \|\varphi_2\|_C) \|\varphi_1 - \varphi_2\|_C,$$

where $L(t, y_1, y_2)$ is continuous with respect to $t \in R_+$, $y_1, y_2 \in R_+$, and is monotonically nondecreasing with respect to y_1, y_2.

Problem (1.1) is related to the integral equation

$$\begin{cases} u(t) = T(t)\varphi(0) + \displaystyle\int_0^t T(t-s) f(s, u_s)\, ds, & t \ge 0, \\ u(t) = \varphi(t), & t \in [-r, 0]. \end{cases} \tag{1.2}$$

The continuous solutions of (1.2) are called mild solutions of (1.1). It should be pointed out that (1.2) is not equivalent to (1.1), i.e., the continuous solutions of (1.2) do not satisfy (1.1) in general. On the other hand, solutions of (1.1) are obviously solutions of (1.2). The *solution* referred to in this article represents the mild solution.

2. EXISTENCE AND STABILITY OF SOLUTIONS

In general, we need to demonstrate the existence of global solutions before we discuss the stability of these solutions. But in this article, we solve these two problems simultaneously. For this purpose, we present the further assumptions:

(H_3) $\|T(t)\| \leq Ke^{-\omega t}$, $K > 0$, $\omega > 0$;

(H_4) (i) there exists a function $G \in C(R_+ \times R_+, R_+)$, monotonically nondecreasing with respect to its second argument, and such that

$$\| f(t, \varphi) \|_X \leq G(t, \|\varphi\|_C), \qquad (t, \varphi) \in R_+ \times C;$$

(ii) there exists $g \in C([-r, \infty) \times R_+, R_+)$ such that for each $p \geq 0$, $g(t) := g(t, p)$ satisfies

$$\begin{cases} g(t) \geq Kpe^{-\omega t} + \int_0^t Ke^{-\omega(t-s)} G(s, g_s)\, ds, & t \geq 0, \\ g(t) \geq Kp, & t \in [-r, 0], \quad K \geq 1, \end{cases}$$

where $g_t = \sup_{-r \leq \theta \leq 0} g(t + \theta)$.

Now we start the iteration procedure. We choose $u^{(0)} \in C([-r, \infty), X)$ so that $\|u_t^{(0)}\|_C \leq g_t$ for $t \geq 0$, where $g(t) = g(t, \|\varphi\|_C)$. For example, we choose $u^{(0)}(t) = \varphi(t)$, $-r \leq t \leq 0$; $u^{(0)}(t) = T(t)\varphi(0)$, $t \geq 0$. It is obvious that $\|u^{(0)}(t)\|_X \leq g_t \in [-r, \infty)$, thus $\|u_t^{(0)}\|_C \leq g_t$. Let

$$\begin{cases} u^{(1)}(t) = T(t)\varphi(0) + \int_0^t T(t-s) f\left(s, u_s^{(0)}\right) ds, & t \geq 0, \\ u^{(1)}(t) = \varphi(t), & t \in [-r, 0]. \end{cases}$$

In general, we define the kth iteration as

$$\begin{cases} u^{(k)}(t) = T(t)\varphi(0) \\ \qquad\qquad + \int_0^t T(t-s) f\left(s, u_s^{(k-1)}\right) ds, & t \geq 0, \\ u^{(k)}(t) = \varphi(t), & t \in [-r, 0], \quad k = 1, 2, \dots . \end{cases} \tag{2.1}$$

Let $k = 1$, from (2.1) with respect to $t \geq 0$, we have

$$\begin{aligned} \left\| u^{(1)}(t) \right\|_X &\leq K \|\varphi(0)\|_X e^{-\omega t} + \int_0^t Ke^{-\omega(t-s)} \left\| f\left(s, u_s^{(0)}\right) \right\|_X ds \\ &\leq Ke^{-\omega t} \|\varphi\|_C + \int_0^t Ke^{-\omega(t-s)} G\left(s, \left\| u_s^{(0)} \right\|_C\right) ds \\ &\leq Ke^{-\omega t} \|\varphi\|_C + \int_0^t Ke^{-\omega(t-s)} G(s, g_s)\, ds \leq g(t). \end{aligned}$$

With respect to $t \in [-r, 0]$, it follows that

$$\left\| u^{(1)}(t) \right\|_X \le \|\varphi\|_C \le K\|\varphi\|_C .$$

Hence we obtain

$$\left\| u^{(1)}(t) \right\|_X \le g(t),\, t \in [-r, \infty) \qquad \text{or} \qquad \|u_t^{(1)}\|_C \le g_t,\, t \ge 0.$$

By using induction, we can prove that for ay positive integer k, the following estimation holds

$$\begin{gathered} \left\| u^{(k)}(t) \right\|_X \le g(t),\, t \in [-r, \infty) \qquad \text{or} \\ \left\| u_t^{(k)} \right\|_C \le g_t,\, t \ge 0,\, k = 1, 2, \dots . \end{gathered} \tag{2.2}$$

Then we prove that for any fixed $\tilde{T} > 0$, the sequence $\{u^{(k)}(t)\}$ is uniformly convergent on $[0, \tilde{T}]$. From (2.1), for $t \in [0, \tilde{T}]$, it is easy to see that

$$\begin{aligned} \left\| u^{(1)}(t) - u^{(0)}(t) \right\|_X &\le \int_0^t K e^{-\omega(t-s)} L\big(s, \|u_s^{(0)}\|_C, 0\big) \|u_s^{(0)}\|_C \, ds \\ &\le \int_0^t K e^{-\omega(t-s)} L(s, g_s, g_s) g_s \, ds. \end{aligned}$$

Noticing g and L are both continuous, we can let $g_t \le N$, $KL(t, g_t, g_t) \le M$, thus we have

$$\|u^{(1)} - u^{(0)}\|_X \le MNt,$$

and it is obvious that

$$\|u_t^{(1)} - u_t^{(0)}\|_C \le MNt.$$

Applying the induction, it is easy to prove that for any positive integer k, the following inequalities hold

$$\begin{aligned} \|u^{(k)} - u^{(k-1)}\|_X &\le N \frac{(M\tilde{T})^k}{k!}, \qquad t \in [0, \tilde{T}], \\ \|u_t^{(k)} - u_t^{(k-1)}\|_C &\le N \frac{(M\tilde{T})^k}{k!}, \qquad t \in [0, \tilde{T}]. \end{aligned} \tag{2.3}$$

Hence $\lim_{k\to\infty} u^{(k)}(t)$ uniformly exists on $[0, \tilde{T}]$. Let $\lim_{k\to\infty} u^{(k)}(t) = u(t)$, then $u(t)$ is continuous on $[0, \tilde{T}]$. Noticing

$$\begin{aligned}
&\left\| u(t) - T(t)\varphi(0) - \int_0^t T(t-s) f(s, u_s)\, ds \right\|_X \\
&\quad \le \left\| u(t) - u^{(k+1)}(t) \right\|_X \\
&\qquad + \left\| \int_0^t T(t-s)\big(f(s,u_s) - f\big(s, u_s^{(k)}\big)\big)\, ds \right\|_X \\
&\quad \le N(1+Mt) \sum_{n=k+1}^{\infty} \frac{M^n t^n}{n!},
\end{aligned}$$

it implies that $u(t)$ satisfies the integral equation (1.2), i.e., $u(t)$ is the mild solution of (1.1).

Finally, we prove that $u(t)$ is the unique one. Let $v(t)$ be another continuous solution of (1.2) and $\|v_t\|_C \le g_t$. It follows that

$$\begin{aligned}
\|u - v\|_X &\le \int_0^t K e^{-\omega(t-s)} L\big(s, \|u_s\|_C, \|v_s\|_C\big) \|u_s - v_s\|_C\, ds \\
&\le \int_0^t K e^{-\omega(t-s)} L(s, g_s, g_s) \|u_s - v_s\|_C\, ds \\
&\le M \int_0^t \|u_s - v_s\|_C\, ds.
\end{aligned}$$

Thus we obtain

$$\|u_t - v_t\|_C \le M \int_0^t \|u_s - v_s\|_C\, ds.$$

By means of the Gronwall inequality, we can get $u(t) \equiv v(t)$ at once.

Summarizing the above discussions, we have

THEOREM 2.1. *If* (H_1)–(H_4) *are satisfied, then the problem* (1.2) *has a unique global continuous solution, i.e., the problem* (1.1) *has a unique global mild solution* $u(t,\varphi)\colon [-r,\infty) \to X$, *and the following inequality holds*

$$\|u(t,\varphi)\|_X \le g(t, \|\varphi\|_C), \qquad t \in [-r, \infty). \tag{2.4}$$

THEOREM 2.2. *Under the conditions of Theorem* 2.1, *and if* $\lim_{p\to 0} \sup_{t\ge 0} g(t,p) = 0$, *then the trivial solution of the problem* (1.1) *is stable. If, furthermore,* $\lim_{t\to\infty} g(t,p) = 0$ *for small* $p > 0$, *then the trivial solution of* (1.1) *is asymptotically stable.*

This conclusion can be deduced directly from the inequality (2.4).

For the convenience of applications, we establish several similar theorems for fractional power spaces. For this purpose, we give the following assumptions replacing the conditions (H_1)–(H_4):

(H'_1) A is the infinitesimal generator of the analytic semigroup $T(t)$ on X which is compact for every $t > 0$;

(H'_2) $\|T(t)\| \le Ke^{-\omega t}$, $K > 0$, $\omega > 0$, $t \ge 0$.

Let $C_\alpha = C([-r,0], X_\alpha)$, for $\varphi \in C_\alpha$, we denote $\|\varphi\|_{C_\alpha} = \sup_{s\in[-r,0]} \|A^\alpha \varphi(\theta)\|_X$.

(H'_3) f: $R_+ \times C_\alpha \to X$ is continuous, $f(t,0) \equiv 0$, and it satisfies the local Lipschitz condition with respect to φ, i.e.,

$$\|f(t,\varphi_1) - f(t,\varphi_2)\|_X \le L\big(t, \|\varphi_1\|_{C_\alpha}, \|\varphi_2\|_{C_\alpha}\big)\|\varphi_1 - \varphi_2\|_{C_\alpha},$$

where $L \in C(R_+ \times R_+ \times R_+, R_+)$ and is monotonically nondecreasing with respect to the second and third arguments;

(H'_4) (i)$'$ there exists $G \in C(R_+ \times R_+, R_+)$ which is monotonically nondecreasing with respect to the second argument such that

$$\|f(t,\varphi)\|_X \le G\big(t, \|\varphi\|_{C_\alpha}\big);$$

(ii)$'$ there exists $g \in C([-r,\infty) \times R_+, R_+)$ such that for every $p \ge 0$, $g(t) := g(t,p)$ satisfies

$$\begin{cases} g(t) \ge Ke^{-\omega t}\|\varphi\|_{C_\alpha} + \int_0^t K(t-s)^{-\alpha} e^{-\omega(t-s)} G(s, g_s)\, ds, & t \ge 0, \\ g_t \ge K\|\varphi\|_{C_\alpha}, & t \in [-r,0]. \end{cases}$$

To apply the fixed point theorem, we define the set $S = \{u \mid u \in C([-r,\infty), X_\alpha), u_0 = \varphi, \|u_t\|_{C_\alpha} \le g_t, t \ge 0\}$. And we define the mapping F on S:

$$\begin{cases} (Fu)(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(s,u_s)\, ds, & t \ge 0 \\ (Fu)(t) = \varphi(t), & t \in [-r,0]. \end{cases} \tag{2.5}$$

First, from (2.5) with respect to $t \ge 0$, we have

$$\begin{aligned}
\|(Fu)(t)\|_\alpha &\le \|T(t)\|\,\|\varphi(0)\|_{X_\alpha} + \left\|\int_0^t A^\alpha T(t-s)f(s,u_s)\,ds\right\|_X \\
&\le Ke^{-\omega t}\|\varphi\|_{C_\alpha} + \int_0^t K(t-s)^{-\alpha}e^{-\omega(t-s)}\|f(s,u_s)\|_X\,ds \\
&\le Ke^{-\omega t}\|\varphi\|_{C_\alpha} + \int_0^t K(t-s)^{-\alpha}e^{-\omega(t-s)}G\big(s,\|u_s\|_{C_\alpha}\big)\,ds \\
&\le Ke^{-\omega t}\|\varphi\|_{C_\alpha} + \int_0^t K(t-s)^{-\alpha}e^{-\omega(t-s)}G(s,g_s)\,ds \\
&\le g(t).
\end{aligned}$$

With respect to $t \in [-r, 0]$, it follows that

$$\|(Fu)(t)\|_\alpha \le \|\varphi\|_\alpha \le \|\varphi\|_{C_\alpha} \le K\|\varphi\|_{C_\alpha} \le g.$$

Then we can get

$$\|(Fu)(t)\|_\alpha \le g\big(t, \|\varphi\|_{C_\alpha}\big), \qquad t \in [-r,\infty).$$

It implies that

$$\|(Fu)_t\|_{C_\alpha} \le g_t, \qquad t \ge 0.$$

Thus $F\colon S \to S$.

Second, for $0 \le t_1 < t_2$, from (2.5) we have

$$\begin{aligned}
&\|(Fu)(t_1) - (Fu)(t_2)\|_\alpha \\
&\quad\le \|T(t_1) - T(t_2)\|\,\|\varphi\|_{C_\alpha} \\
&\qquad + \left\|\int_0^{t_1} A^\alpha T(t_1-s)f(s,u_s)\,ds - \int_0^{t_2} A^\alpha T(t_2-s)f(s,u_s)\,ds\right\|_X \\
&\quad\le \|T(t_1) - T(t_2)\|\,\|\varphi\|_{C_\alpha} \\
&\qquad + \left\|\int_0^{t_1} A^\alpha\big(T(t_1-s) - T(t_2-s)\big)f(s,u_s)\,ds\right\|_X \\
&\qquad + \left\|\int_{t_1}^{t_2} A^\alpha T(t_2-s)f(s,u_s)\,ds\right\|_X \\
&\quad\le \|T(t_1) - T(t_2)\|\,\|\varphi\|_{C_\alpha} + \|T(\varepsilon) - T(t_2 - t_1 + \varepsilon)\|
\end{aligned}$$

$$\times \left\| \int_0^{t_1-\varepsilon} A^\alpha T(t_1-s-\varepsilon) f(s,u_s)\,ds \right\|_X$$

$$+ \left\| \int_{t_1-\varepsilon}^{t_1} A^\alpha (T(t_1-s)-T(t_2-s)) f(s,u_s)\,ds \right\|_X$$

$$+ KM_f \int_0^{t_2-t_1} s^{-\alpha} e^{-\omega s}\,ds$$

$$\le \| T(t_1)-T(t_2) \| \, \|\varphi\|_{C_\alpha}$$

$$+ \| T(\varepsilon) - T(t_2-t_1+\varepsilon) \| KM_f \int_0^{t_1-\varepsilon} s^{-\alpha} e^{-\omega s}\,ds$$

$$+ KM_f \left[\int_0^{\varepsilon} s^{-\alpha} e^{-\omega s}\,ds + \int_0^{t_2-t_1+\varepsilon} s^{-\alpha} e^{-\omega s}\,ds \right]$$

$$+ 2KM_f \int_0^{t_2-t_1} s^{-\alpha} e^{-\omega s}\,ds.$$

For any fixed $\tilde{T} > 0$, from the compactness of $T(t)$ we know that $(Fu)(t)$ is equicontinuous on $[0, \tilde{T}]$. So the set $\{(Fu)(t), u \in S\}$ is equicontinuous.

Third, $\overline{FS}$ is compact. For this purpose, we only need to prove that for fixed $t \in [-r, \tilde{T}]$, $\{(Fu)(t), u \in S\}$ is relatively compact. Noticing for $0 \le \alpha < \beta < 1$, when $t \in [0, \tilde{T}]$, it follows that

$$\| A^\beta (Fu)(t) \|_X \le \| T(t) \| \, \|\varphi\|_{C\beta} + \left\| \int_0^t A^\beta T(t-s) f(s,u_s)\,ds \right\|_X$$

$$\le Ke^{-\omega t} \|\varphi\|_{C\beta} + KM_f \int_0^t e^{-\omega s} s^{-\alpha}\,ds.$$

When $t \in [-r, 0]$, we have

$$\| A^\beta (Fu)(t) \|_X \le \|\varphi\|_\beta \le \|\varphi\|_{C_\beta}.$$

It shows that $\{A^\beta (Fu)(t)\}$ is bounded in X, thus $A^{-\beta}$: $X \to X_\alpha$ is compact.

Fourth, the mapping F is continuous. From the continuity of f, for given $\varepsilon > 0$, there exists $\delta > 0$, when $\sup \|u(s) - \hat{u}(s)\|_\alpha < \delta$, such that

$$\sup_{0 \le s \le \tilde{T}} \| f(s,u_s) - f(s,\hat{u}_s) \|_X < \varepsilon.$$

Then with respect to $t \in [0, \tilde{T}]$, $\sup \|u(s) - \hat{u}(s)\|_\alpha < \delta$, we have

$$\begin{aligned}
&\|(Fu)(t) - (F\hat{u})(t)\|_\alpha \\
&\qquad \le \left\| \int_0^t A^\alpha T(t-s)\big(f(s,u_s) - f(s,\hat{u}_s)\big)\, ds \right\|_X \\
&\qquad \le \int_0^t K(t-s)^{-\alpha} e^{-\omega(t-s)} L\big(s, \|u_s\|_{C_\alpha}, \|\hat{u}_s\|_{C_\alpha}\big) \|u_s - \hat{u}_s\|_{C_\alpha}\, ds \\
&\qquad \le \varepsilon M \int_0^t s^{-\alpha} e^{-\omega s}\, ds.
\end{aligned}$$

So F is continuous.

According to Schauder fixed point theorem, F has a fixed point in S, i.e., the problem (1.1) has a global mild solution $u \in C([-r,\infty), X_\alpha)$ in X_α which satisfies the inequality

$$\|u(t,\varphi)\|_\alpha \le g\big(t, \|\varphi\|_{C_\alpha}\big), \qquad t \in [-r, \infty). \tag{2.6}$$

Finally, we prove the uniqueness. Suppose $v(t,\varphi)$ is another continuous solution of (1.2), $v \in S$. We choose $\tau > 0$ small enough so that $M\int_0^\tau s^{-\alpha} e^{-\omega s}\, ds < 1/2$. Then

$$\begin{aligned}
&\|u(t,\varphi) - v(t,\varphi)\|_\alpha \\
&\qquad \le \left\| \int_0^t A^\alpha T(t-s)\big(f(s,u_s) - f(s,v_s)\big)\, ds \right\|_X \\
&\qquad \le \int_0^t K(t-s)^{-\alpha} e^{-\omega(t-s)} L\big(s, \|u_s\|_{C_\alpha}, \|v_s\|_{C_\alpha}\big) \|u_s - v_s\|_{C_\alpha}\, ds \\
&\qquad \le \int_0^t K(t-s)^{-\alpha} e^{-\omega(t-s)} L(s, g_s, g_s) \sup_{0 \le t \le r} \|u_t - v_t\|_{C_\alpha}\, ds \\
&\qquad \le M \int_0^\tau s^{-\alpha} e^{-\omega s}\, ds \cdot \sup_{0 \le t \le \tau} \|u_t - v_t\|_{C_\alpha} \\
&\qquad < \frac{1}{2} \sup_{0 \le t \le \tau} \|u_t - v_t\|_{C_\alpha}.
\end{aligned}$$

Thus we obtain

$$\sup_{0 \le t \le \tau} \|u_t - v_t\|_{C_\alpha} \le \frac{1}{2} \sup_{0 \le t \le \tau} \|u_t - v_t\|_{C_\alpha},$$

so it implies that $u(t,\varphi)=v(t,\varphi)$, $t\in[0,\tau]$. Repeating this procedure, we can get $u(t,\varphi)=v(t,\varphi)$ for $t\in R_+$.

Summarizing the above proofs, we have

THEOREM 2.3. *If* (H'_1)–(H'_4) *are satisfied, then problem* (1.1) *has a unique global mild solution* $u\in C([-r,\infty),X_\alpha)$, *and the inequality* (2.6) *holds*.

THEOREM 2.4. *Under the conditions of Theorem* 2.3, *and if* $\lim_{p\to 0}\sup_{t\geq 0} g(t,p)=0$, *then the trivial solution of problem* (1.1) *is stable*. *If, furthermore*, $\lim_{t\to\infty} g(t,p)=0$ *for small* $p>0$, *then the trivial solution of* (1.1) *is asymptotically stable*.

This conclusion can be obtained from the inequality (2.6).

3. APPLICATIONS

In this section, we will apply the above results to two known problems. The first one is a generalization of a corresponding problem in [6]. The second one can be found in [4].

1. We discuss the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}=A(x,D)u+F(t,x,u_r,\nabla u_r), & (t,x)\in R_+\times\Omega,\\ u(t,x)=0, & (t,x)\in R_+\times\partial\Omega,\\ u(t,x)=\varphi(t,x), & (t,x)\in[-r,0]\times\overline{\Omega}, \end{cases} \tag{3.1}$$

where $u_r=u(t-r,x)$ $(r>0)$, Ω is a bounded domain in R^3 with the smooth boundary $\partial\Omega$ and

$$A(x,D)=\sum_{k,l}^{3}\frac{\partial}{\partial x_k}\left(a_{k,l}(x)\frac{\partial}{\partial x_l}\right)$$

is a uniform elliptic operator, $a_{k,l}=a_{l,k}$ $(k,l=1,2,3)$ is continuously differentiable on $\overline{\Omega}$, and F is continuous with respect to all variables.

We further suppose that there exists a continuous function $K(t,y)$: $R_+\times R_+\to R_+$ which is increasing with respect to the second argument and satisfies the inequalities

$$\|F(t,x,u_r,p_r)\|\leq K(t,\|u_r\|)(1+\|p_r\|^{\gamma}),$$

$$\|F(t,x,u_r,p_r)-F(t,x,u_r,q_r)\|$$

$$\leq K\big(t, \|u_r\|\big)\big(1 + \|p_r\| + \|q_r\|\big)\|p_r - q_r\|,$$

$$\|F(t,x,u_r,p_r) - F(t,x,v_r,p_r)\|$$

$$\leq K\big(t, \|u_r\| + \|v_r\|\big)\big(1 + \|p_r\|^{\gamma}\big)\|u_r - v_r\|,$$

for $(t,x,u_r,p_r),(t,x,u_r,q_r),(t,x,v_r,p_r) \in R_+ \times \Omega \times R \times R^3$, $1 \leq \gamma < 3$.

Let $A = A(x,D)$ be the linear operator in L_2, $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, $Au = A(x,D)u$, for $u \in D(A)$. Denote $f(t,\varphi)(x) = F(t, x, \varphi(-r)(x), \nabla\varphi(-r)(x))$, then (3.1) can be written as

$$\begin{cases} \dfrac{du}{dt} = Au + f(t,u_r), & t \geq 0, \\ u = \varphi, & t \in [-r,0]. \end{cases} \tag{3.2}$$

According to [6, Theorem 7.3.6], A is the infinitesimal generator of an analytic semigroup in $L_2(\Omega)$, and it is easy to prove that $T(t)$ is compact with respect to every $t > 0$, $\|T(t)\| \leq Ke^{-\omega t}$, $\omega > 0$. Thus the conditions (H_1'), (H_2') hold.

(1) We claim that $f(t,\varphi)$ has definition on $R_+ \times C_\alpha$. Noticing if $\alpha > 3/4$, then $X_\alpha \subset L_\infty(\Omega)$. And if $1/q > (5-4\alpha)/6$, then $X_\alpha \subset W^{1,q}(\Omega)$. Then for $\max\{3/4, (5\gamma-3)/4\gamma\} < \alpha < 1$, it follows that

$$X_\alpha \subset W^{1,2}(\Omega) \cap L_\infty(\Omega).$$

Thus for $\varphi \in C_\alpha$ and applying the embedding theorem, we can get

$$\begin{aligned} \|f(t,\varphi)\|_{L_2} &= \|F(t,x,\varphi(-r)(x), \nabla\varphi(-r)(x))\|_{L_2} \\ &\leq K\big(t, \|\varphi(-r)\|_{L_\infty}\big)\big(\Omega^{1/2} + \|\varphi(-r)\|_{1,2}^{\gamma}\big) \\ &\leq K\big(t, \|\varphi(-r)\|_{\alpha}\big)\big(\Omega^{1/2} + \|\varphi(-r)\|_{\alpha}^{\gamma}\big) \\ &\leq K\big(t, \|\varphi\|_{C_\alpha}\big)\big(\Omega^{1/2} + \|\varphi\|_{C_\alpha}^{\gamma}\big). \end{aligned} \tag{3.3}$$

So $f(t,\varphi)$ has definition on $R_+ \times C_\alpha$.

(2) We claim that $f(t,\varphi)$ is locally Lipschitz continuous with respect to φ. Suppose $\varphi, \psi \in C_\alpha$, it follows that

$$\begin{aligned} &\|f(t,\varphi) - f(t,\psi)\|_{L_2}^2 \\ &\quad = \int_\Omega \big|F(t,x,\varphi(-r)(x), \nabla\varphi(-r)(x)) \\ &\qquad - F(t,x,\psi(-r)(x), \nabla\psi(-r)(x))\big|^2\, dx \end{aligned}$$

$$
\begin{aligned}
&\leq 2\int_\Omega |F(t,x,\varphi(-r)(x),\nabla\varphi(-r)(x)) \\
&\qquad -F(t,x,\psi(-r)(x),\nabla\varphi(-r)(x))|^2\,dx \\
&\quad + 2\int_\Omega |F(t,x,\psi(-r)(x),\nabla\varphi(-r)(x)) \\
&\qquad\quad -F(t,x,\psi(-r)(x),\nabla\psi(-r)(x))|^2\,dx \\
&\leq 2K^2\big(t,\|\varphi(-r)\|_{L_\infty}+\|\psi\|_{L_\infty}\big) \\
&\quad \times\int_\Omega (1+|\nabla\varphi(-r)|)^2|\varphi(-r)-\psi(-r)|^2\,dx \\
&\quad + 2K^2\big(t,\|\varphi(-r)\|_{L_\infty}\big) \\
&\qquad \times\int_\Omega \big(1+|\nabla\varphi(-r)|^{\gamma-1}+|\nabla\psi(-r)|^{\gamma-1}\big)^2|\nabla\varphi(-r)-\nabla\psi(-r)|^2\,dx \\
&\leq C_1K^2(t,\|\varphi(-r)\|_\alpha+\|\psi(-r)\|_\alpha)\big(1+\|\nabla\varphi(-r)\|_{L_2}^{2\gamma}\big) \\
&\quad \times\int_\Omega |\nabla\varphi(-r)-\nabla\psi(-r)|^2\,dx \\
&\quad + C_2K^2(t,\|\varphi(-r)\|_\alpha)\big(1+\|\nabla\varphi(-r)\|_{L_2}^{2(\gamma-1)}+\|\nabla\psi(-r)\|_{L_2}^{2(\gamma-1)}\big) \\
&\qquad \times\int_\Omega |\nabla\varphi(-r)-\nabla\psi(-r)|^2\,dx \\
&\leq L\big(t,\|\varphi\|_{C_\alpha},\|\psi\|_{C_\alpha}\big)\|\varphi-\psi\|_{C_\alpha}^2.
\end{aligned}
$$

So $f(t,\varphi)$ is locally Lipschitz continuous. Hence the condition (H_3') holds.

From the above proofs, we have

THEOREM 3.1. *Suppose that* $K(t,y) = b(1+t^{1-\alpha})^{-\sigma}y^\sigma$ $(\sigma > 1)$, $\|\varphi\|_{C_\alpha}$ *is small enough. Then the global mild solution of problem* (3.2) *exists, and the trivial solution is asymptotic stable in* X_α.

Proof. We only need to prove that the condition (H_4') is satisfied. For this purpose, we choose

$$
\begin{aligned}
G(t,y) &= d(1+t^{1-\alpha})^{-\sigma}y^\sigma(1+y^\gamma), \\
g(t,p) &= Kp\big[1+(t+r)^{1-\alpha}\big]e^{-\omega t}.
\end{aligned}
$$

From (3.3), we know the inequality (i$'$) obviously holds with respect to G.

When $t\in[-r,0]$, it is obvious that $g(t)\geq K\|\varphi\|_{C_\alpha}$, where $g(t) := g(t,\|\varphi\|_{C_\alpha})$.

When $t \geq 0$, it follows that

$$
\begin{aligned}
I &= g(t) - e^{-\omega t}K\|\varphi\|_{C_\alpha} \\
&\quad - \int_0^t K(t-s)^{-\alpha} e^{-\omega(t-s)} d(1+s^{1-\alpha}) g_s^\sigma (1+g_s^\gamma)\, ds \\
&= e^{-\omega t}K\|\varphi\|_{C_\alpha} t^{1-\alpha} - e^{-\omega t}K^{1+\sigma}\, de^{\omega\tau}\|\varphi\|_{C_\alpha}^\sigma \\
&\quad \times \int_0^t (t-s)^{-\alpha} e^{-\omega(\sigma-1)s}\left[1 + K^\gamma\|\varphi\|_{C_\alpha}^\gamma (1+s^{1-\alpha}) e^{-\omega\gamma s}\right] ds.
\end{aligned}
$$

Noticing for $a \geq 0$, $b \geq 0$, $n \geq 1$, we have

$$
(a+b)^n \leq 2^{n-1}(a^n + b^n). \tag{3.4}
$$

Then we can get

$$
\begin{aligned}
I &\geq e^{-\omega t}K\|\varphi\|_{C_\alpha} t^{1-\alpha} - e^{-\omega t}K^{1+\alpha}\, de^{\omega\tau}\|\varphi\|_{C_n}^\sigma \\
&\quad \times \left\{ \int_0^t (t-s)^{-\alpha}\, ds + 2^\gamma \int_0^t (t-s)^{-\alpha} K^\gamma \|\varphi\|_{C_\alpha}^\gamma (1+s^{(1-\alpha)\gamma}) e^{-\omega\gamma s}\, ds \right\} \\
&\geq e^{-\omega t}K\|\varphi\|_{C_\alpha} t^{1-\alpha} - e^{-\omega t}K^{1+\sigma}\, de^{\omega\tau}\|\varphi\|_{C_\alpha}^\gamma\Big) \\
&\quad \times \left\{ \int_0^t \left(1 + 2^\gamma K^\gamma \|\varphi\|_{C_\alpha}^\gamma\right)(t-s)^{-\alpha}\, ds \right. \\
&\qquad \left. + \int_0^t K^\gamma \|\varphi\|_{C_\alpha}^\gamma (t-s)^{-\alpha} s^{(1-\alpha)\gamma} e^{-\omega\gamma s}\, ds \right\}.
\end{aligned}
$$

Let $h(s) = s^{(1-\alpha)\gamma} e^{-\omega\gamma s}$, it is easy to see that $h(s)$ attains its maximum value M^* at $s = (1-\alpha)/\omega$. So we obtain

$$
\begin{aligned}
I &\geq e^{-\omega t}K\|\varphi\|_{C_\alpha} t^{1-\alpha} \\
&\quad \times \left\{ 1 - K^\sigma \|\varphi\|_{C_\alpha}^{\sigma-1}\, de^{\omega\tau}\left[1 + \alpha^\gamma K^\gamma \|\varphi\|_{C_\alpha}^\gamma (1+M^*) \frac{1}{1-\alpha}\right]\right\}.
\end{aligned}
$$

Since $\|\varphi\|_{C_\alpha}$ is small enough and $\sigma > 1$, it follows that $I \geq 0$. Thus the condition (H_4') holds. According to Theorem 2.3, the problem (3.2) has a unique global mild solution $u \in C([-r,\infty), X_\alpha)$, and the following inequality

$$
\|u(t,\varphi)\|_\alpha \leq g\left(t, \|\varphi\|_{C_\alpha}\right) = K\|\varphi\|_{C_\alpha}\left[1 + (t+\tau)^{1-\alpha}\right] e^{-\omega t}
$$

holds. It is obvious that $g \to 0$ as $t \to \infty$. Then the theorem is proved.

Remark 1. The above results at least have two novel and original ideas in contrast to the corresponding results of [6]:

(1^0) The equation studied in [6] is a partial differential equation, but the equation discussed in this article is a partial differential equation with time delay. There is no doubt that the latter one is more complex than the former, and it is also more difficult to deal with.

(2^0) The solution obtained in [6] is a local one (strong), but here we obtain the global solution (mild). Furthermore, we obtain the asymptotic stability of the trivial solution if the $K(t, y)$ are given the appropriate form, and the initial value $\|\varphi\|_{C_\alpha}$ is sufficiently small.

2. We discuss the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - u^3(t-\tau, x), & (t, x) \in R_+ \times \Omega, \\ u(t, x) = 0, & (t, x) \in R_+ \times \partial\Omega, \\ u(t, x) = \varphi(t, x), & (t, x) \in [-r, 0] \times \overline{\Omega}, \end{cases} \tag{3.5}$$

where Ω is a bounded domain in R^3 with the smooth boundary $\partial\Omega$. As we did in the first example, let $A = \Delta$ be the linear operator in $L_2(\Omega)$, $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, $Au = \Delta u$, $u \in D(A)$. We define $f(\varphi)(x) = -\varphi^3(-r)(x)$, then (3.5) can be written as

$$\begin{cases} \dfrac{du}{dt} = Au + f(u_r), & t \geq 0, \\ u = \varphi(t), & t \in [-r, 0]. \end{cases} \tag{3.6}$$

From the problem (3.1), it is easy to know that A is the infinitesimal generator of an analytic semigroup $T(t)$ on $L_2(\Omega)$ which is compact with respect to every $t > 0$, and $\|T(t)\| \leq Ke^{-\omega t}$, $\omega > 0$, thus (H_1') and (H_2') hold.

Choose $\alpha = 1/2$.

(1) We claim that $f(\varphi)$ has definition on $R_+ \times C_{1/2}$. Noticing $L_6(\Omega) \subset H_0^1(\Omega)$, we have

$$\| f(\varphi) \|_{L_2}^2 = \int_\Omega \left| \varphi^3(-r)(x) \right|^2 dx = \| \varphi(-r) \|_{L_6}^6.$$

From the above expression and applying the embedding theorem, we can get

$$\| f(\varphi) \|_{L_2} = \| \varphi(-r) \|_{L_6}^3 \leq C \| \varphi(-r) \|_{H_0^1}^3 \leq C \|\varphi\|_{C_{1/2}}^3. \tag{3.7}$$

(2) We claim that $f(\varphi)$ is locally Lipschitz continuous. Making use of the Hölder inequality and the embedding theorem, we obtain

$$\begin{aligned}
&\|f(\varphi)-f(\psi)\|_{L_2}^2\\
&\quad=\int_\Omega\left|\varphi^3(-r)-\psi^3(-r)\right|^2dx\\
&\quad\le\int_\Omega\left|\varphi^2(-r)+\varphi(-r)\psi(-r)+\psi^2(-r)\right|^2\\
&\qquad\times\left|\varphi(-r)-\psi(-r)\right|^2dx\\
&\quad\le 2\left\{\left(\int_\Omega|\varphi(-r)|^6\,dx\right)^{2/3}+\left(\int_\Omega|\psi(-r)|^6\,dx\right)^{2/3}\right\}\\
&\qquad\times\left(\int_\Omega|\varphi(-r)-\psi(-r)|^6\,dx\right)^{1/3}\\
&\quad\le 2\Big(\|\varphi(-r)\|_{L_6}^4+\|\psi(-r)\|_{L_6}^4\Big)\|\varphi(-r)-\psi(-r)\|_{L_6}^2\\
&\quad\le C\Big(\|\varphi(-r)\|_{H_0^1}^4+\|\psi(-r)\|_{H_0^1}^4\Big)\|\varphi(-r)-\psi(-r)\|_{H_0^1}^2\\
&\quad\le L\Big(\|\varphi\|_{C_{1/2}},\|\psi\|_{C_{1/2}}\Big)\|\varphi-\psi\|_{C_{1/2}}^2.
\end{aligned}$$

It implies that $f(\varphi)$ is locally Lipschitz continuous with respect to $\varphi\in C_{1/2}$. Hence the condition (H_3') holds.

(3) By choosing $G=dy^3$ $(d\ge C)$, $g(t,p)=Kp[1+(t+r)^{1/2}]e^{-\omega t}$, we prove that the condition (H_4') holds.

From (3.7) we know (i)$'$ holds.

When $t\in[-r,0]$, it is obvious that $g(t):=g(t,\|\varphi\|_{C_{1/2}})\ge K\|\varphi\|_{C_{1/2}}$. And when $t\ge 0$, it follows that

$$\begin{aligned}
I&=g(t)-e^{-\omega t}K\|\varphi\|_{C_{1/2}}-\int_0^t K(t-s)^{-1/2}e^{-\omega(t-s)}\,dk^3\,\|\varphi\|_{C_{1/2}}^3\\
&\qquad\times(1+s^{1/2})^3e^{-3\omega(s-r)}\,ds\\
&=e^{-\omega t}K\|\varphi\|_{C_{1/2}}(t+r)^{1/2}-e^{-\omega t}K^4d\|\varphi\|_{C_{1/2}}^3e^{3\omega r}\\
&\qquad\times\int_0^t(t-s)^{-1/2}(1+s^{1/2})^3\,ds.
\end{aligned}$$

By using the inequality (3.4) and denoting the maximum value of $h(s) = s^{3/2}e^{-2\omega s}$ as M^*, we can get

$$I \geq e^{-\omega t}K\|\varphi\|_{C_{1/2}}(t+r)^{1/2} - e^{\omega t}K^4 d\|\varphi\|^3_{C_{1/2}}e^{3\omega r}2^3$$
$$\times\left\{\int_0^t (t-s)^{-1/2}\,ds + \int_0^t (t-s)^{-1/2}s^{3/2}e^{-2\omega s}\,ds\right\}$$
$$\geq e^{-\omega t}K\|\varphi\|_{C_{1/2}}t^{1/2}\left\{1 - e^{3\omega r}K^3 d\|\varphi\|^2_{C_{1/2}}2^4(1+M^*)\right\}.$$

It implies that $I \geq 0$ as $\|\varphi\|_{C_{1/2}}$ is small enough. Hence the condition (H_4') holds. So we have

THEOREM 3.2. *Suppose $\varphi \in C([-r,0], H_0^1)$, and if $\|\varphi\|_{C_{1/2}}$ is sufficiently small, then problem* (3.6) *has a unique global mild solution $u \in C([-r,\infty), H_0^1)$. Furthermore, the inequality*

$$\|u(t,\varphi)\|_{H_0^1} \leq K\|\varphi\|_{C_{1/2}}\left[1 + (t+r)^{1/2}\right]e^{-\omega t}, \qquad t \in [-r,\infty)$$

holds, and hence its trivial solution is asymptotically stable in H_0^1.

For $f(\varphi)(x) = -\varphi^2(-r)(x)\varphi(0)(x),\ -\varphi(-r)(x)\varphi^2(0)(x)$, we can also obtain similar results. Since the proof is similar, we will not repeat the procedure.

Remark 2. These results are superior to the corresponding results of [4] in the following ways:

(1^0) Since the structure of the comparison function is very simple (only depends on the initial function and the eigenvalue of the operator A), the conclusion is fairly concrete.

(2^0) The method is very concise, concrete, and unitary.

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